

河內塔之探討

任教科別：數學科 作者：李芳俞 老師

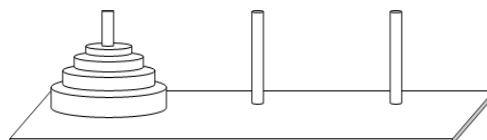
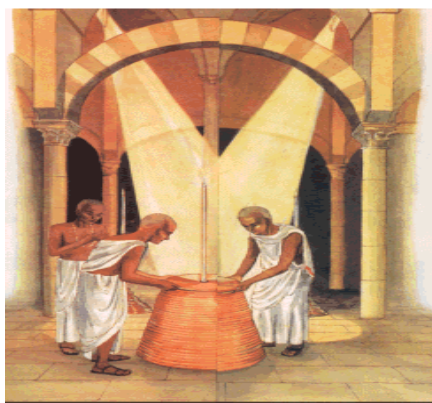
摘要

河內塔是一個經典的數學遊戲，在你我高中時代，它是一個課本中的例題。主要是用來說明遞迴關係。當時，我們較少會真的去實際操作，如此甚為可惜。

期望透過這個活動讓同學學習探索數學的方法，其實與探索自然的方法是差不多的。從實驗、參考文獻、猜測、獲得初步結論，到進一步推廣、修改或驗證結論。最後形成公式、性質、定理或定律。

壹、前言

1883 年，一位法國的數學家 Edouard Lucas 教授在歐洲的一份雜誌上介紹了一個相當吸引人的難題——迷人的智力遊戲。這個遊戲名為河內塔(Tower of Hanoi)，它源自古印度神廟中的一段故事(也有一說是 Lucas 教授為增加此遊戲之神秘色彩而捏造的)。古印度有一個很有趣的傳說，在伯那爾斯(Benares)的一座大寺廟裡，有一棟被稱為是世界之中心點的樓閣，它的下面有一塊黃銅製的方形盤子，盤子上豎立著三根鑽石製成的棒子。在開天闢地的時候，萬能的上帝波羅摩(Brahma)在其中一根鑽石棒上放置了六十四個大小都不相同的金圈，這些金圈是依大在下小在上順序地疊放著，叫河內塔。古印度人也稱它為梵天培(Towers of Brahma)。上帝所頒下的意旨，是要廟裡的婆羅門和尚們將這六十四個金圈移到另一根鑽石棒上，不過，在移動這些金圈時，有兩項規則：第一是每一次只能移動一個金圈；第二是在移動的過程中，較大的金圈不能放在較小的金圈上。直到有一天，和尚們能將 64 片的金屬片依規則從指定的木釘上全部移動至另一根木釘上，那麼，世界末日即隨之來到，世間的一切終將被毀滅，萬物都將至極樂世界。



貳、正文

一、河內塔的數學

透過河內塔的實際操作與演練，可以讓學生很快理解遞迴與疊代法(Iterative Method)的意義。最早發現河內塔問題是法國數學家愛德華·盧卡斯(François Édouard Anatole Lucas, 1842-1891)，如圖一所示，盧卡斯以研究費伯納契數列聞名，他曾給出費伯納契數列第 n 項的表式法，另外，他也曾經嘗試去改寫費伯納契數列的想法，如令數列 $1, 2, 1, 3, 2, 5, 3, 8, 5, 13, 8, 21, 13, 34, 21, 55, 34, 89, 55, 144, 89, 233, 144, 377, 233, 610, 377, 987, 610, 1597, 987, 2584, 1597, 4181, 2584, 5777, 4181, 8945, 5777, 12139, 8945, 18481, 12139, 24476, 18481, 37701, 24476, 46368, 37701, 72776, 46368, 79483, 72776, 121393, 79483, 196418, 121393, 317811, 196418, 514135, 317811, 777876, 514135, 1095713, 777876, 1618033, 1095713, 2618161, 1618033, 3413161, 2618161, 5052138, 3413161, 6765808, 5052138, 8761815, 6765808, 11497523, 8761815, 15141637, 11497523, 19641815, 15141637, 25854101, 19641815, 33496117, 25854101, 44159215, 33496117, 58131364, 44159215, 75474609, 58131364, 99149062, 75474609, 129873397, 99149062, 169792606, 129873397, 223691414, 169792606, 293787123, 223691414, 387613567, 293787123, 509457864, 387613567, 672581491, 509457864, 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就像其他的益智遊戲，對於河內塔遊戲也可以經由數學上的方法一些漂亮的結果：若一開始就考慮 64 片金屬片似乎太難了，得到我們不妨把金屬片的數量降低，看看會有什麼結果？

(1)玩一個圓環，只須要一步。

(3)玩三個圓環，需先完成前面二個環(三步)，再把第三個圓環移動到目的地(一步)，然後再移動當才移動的二個環(三步)，所以需要 $3+1+3=7$ 步。

(5)如此繼續下去玩八個圓環，需先完成前面七個環(127步)，再把第八個圓環移動到目的地(一步)，然後再移動剛才移動的七個環(127步)，所以需要 $127+1+127=255$ 步。

2. 數學知識--指數律的感覺 - 32 個環，你一生搬不完

我們瞭解如何搬動河內塔的圓環之後，再來看看它有什麼特殊的地方。我們搬動八個環所需的時間和步驟不多。假設我們每次搬動一次只花一秒鐘，所需的時間只要 256 秒，大約 5 分鐘。現在我們來看看搬動 32 個環（不多只是 8 的 4 倍而已）所需的步驟和時間。

假設我們每次都不會搬錯，所以我們只要搬 $2^{32}-1$ 次即可。我們來估計一下需要花多少和多少時間。我們知道 $2^{10}=1024$ 大約是 1000。所以， $2^{32}-1=(2^{10})^3 \times 2^2-1$ 次大約需 4,000,000,000 次。

假設我們每次搬動一次只花一秒鐘。我們一天可以搬 $24(\text{時}) \times 60(\text{分}) \times 60(\text{秒})=86400$ 次。就算 100,000 次好了。那我們需要花 $4,000,000,000/100,000=40,000$ 天，大約是 100 年 $(40,000/365)$ 。這也就是說一個人一生絕對搬不完 32 個環的河內塔。

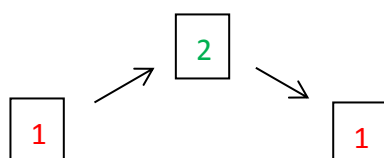
也就是說 64 片金屬片至少要 $=18,446,744,073,709,551,615$ 次的搬移才能完成。假如一位熟練的僧侶每秒中搬移一次金屬片，日以繼夜不眠不休的工作，也需要約 584,942,417,355 年，這是非常非常遙遠的事，科學家估計地球約已存在 2,000,000,000 年，也沒有一種生物能活這麼久，所以我們大可放心的睡覺。

二、河內塔與二進位關係

先把 1 到 2^n-1 的數字寫成一行，再把每個數字的二進制表示法的每一位數寫成一行，最後在最右邊一行寫出對應的二進制數字，就形成一張表格。這張表最左邊的一行，代表移動圓盤的步驟數。現在將圓盤由小至大依序編號為 1, 2, 3, 4, 5, …，並且給每一個圓盤一種不同的顏色。筆者發現在每一移動步驟中，被移動的圓盤那行相應的位置剛好是 1，我們可以用該圓盤的顏色塗那個 1。於是圓盤移動次序與二進位數之間有 1-1(one to one)對應關係，2 透過二進位的數值表徵更容易掌握圓盤的移動情形，那些塗過色的 1 更可以連結成有向路徑圖來表示：

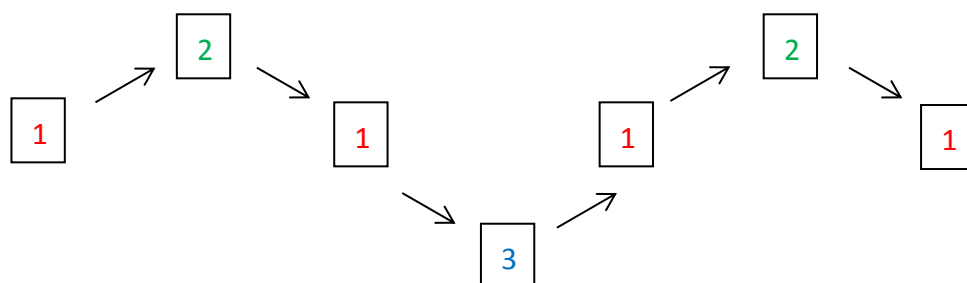
兩個圓盤：圓盤的移動次序為 1 2 1

移動步驟 (十進位數)	圓盤 2	圓盤 1	二進位數
1	0	1	01
2	1	0	10
3	1	1	11



三個圓盤：圓盤的移動次序為為 1 2 1 3 1 2 1

移動步驟 (十進位數)	圓盤 3	圓盤 2	圓盤 1	二進位數
1	0	0	1	001
2	0	1	0	010
3	0	1	1	011
4	1	0	0	100
5	1	0	1	101
6	1	1	0	110
7	1	1	1	111



參、結語

遞迴不僅僅在數學上有其重要性，在電腦科學之中扮演的角色更是至關重要。程式設計者對於遞迴絕對不會陌生，上面所舉的河內塔問題，實際上也是電腦科學的經典例子之一，是初學程式設計的人一定會學到的東西。遞迴的思維，常常可以讓程式設計者打造出簡潔的程式，讓繁冗的問題透過簡單的程式碼來解決(例如 parser 的設計)。演算法上所講的 dynamic programming，就是遞迴思維在演算法的具體呈現。

遞迴同時也是碎形(fractal)這門大學問的基石，碎形是一種相當美妙的幾何圖案，碎形是遞迴在幾何學的一種具體呈現。但是碎形不僅僅是一種數學概念而已，在自然界中，有許許多多的地方都出現自然的碎形，讓人讚嘆遞迴原來就出現在我們的生活周圍。

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